Some special Kähler metrics on $SL(2,\mathbb{C})$ and their holomorphic quantization

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Abstract

The group $SU(2) \times SU(2)$ acts naturally on $SL(2,\mathbb{C})$ by simultaneous right and left multiplication. We study the Kähler metrics invariant under this action using a global Kähler potential. The volume growth and various curvature quantities are then explicitly computable. Examples include metrics of positive, negative and zero Ricci curvature, and the 1-lump metric of the $\mathbb{C}P^1$-model on a sphere.

We then look at the holomorphic quantization of these metrics, where some physically satisfactory results on the dimension of the Hilbert space can be obtained. These give rise to an interesting geometrical conjecture, regarding the dimension of this space for general Stein manifolds in the semi-classical limit.

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Part I

1 Introduction

Among the geometrical procedures for quantization, holomorphic quantization is a particularly simple and natural one, and can be used whenever the classical system "lives" on a complex Kähler manifold. In this paper the complex manifold under study will be $SL(2,\mathbb{C})$, and we will consider the Kähler metrics on this manifold which are invariant under a natural action of the group $SU(2) \times SU(2)$, namely the action defined by simultaneous right and left multiplication of the matrix in $SL(2,\mathbb{C})$ by the matrices in $SU(2)$.

In the first part of the paper a purely classical study of these Kähler metrics is carried out. We find that each of these metrics has a global invariant Kähler potential, which is essentialy unique, and is in fact a function of only one real variable. We then use this potential to compute explicitly several properties of the Kähler manifold. These include the scalar curvature, a potential for the Ricci form, the volume and volume growth, the geodesic distance from the submanifold $SU(2) \subset SL(2,\mathbb{C})$, and a simple criterion for completeness. Choosing particular functions as Kähler potentials we get metrics with positive-definite, negative-definite and zero Ricci tensor; the Ricci-flat one being just the usual Stenzel metric on $T^*S^3 \simeq SL(2,\mathbb{C})$.

A significant application of the above results, which was in fact the original motivation for this paper, is a closer study of the L^2-metric on the moduli space of one lump on a sphere. These lumps are a particular kind of soliton that appear in $\mathbb{C}P^1$-sigma models, and have been widely studied [2, 13]. In particular, the special case of one lump on a sphere has been studied by Speight in [10, 11], where the author also examines general invariant Kähler metrics on $SL(2,\mathbb{C})$ and finds some of the results mentioned above. The approach there however is rather different, since it is based on the choice of a particular frame for $T^*SL(2,\mathbb{C})$, instead of using the perhaps more natural Kähler potentials.

The second part of the paper examines some aspects of holomorphic quantization on the manifold $SL(2,\mathbb{C})$ with the Kähler metrics described above. We basically look at two things: the nature and dimension of the quantum Hilbert space, and the quantum operators corresponding to the classical symmetries of the metric.

Regarding the latter point, we start by finding the moment map of the $SU(2) \times SU(2)$ action. This map encodes the classical symmetries of the system and, through the usual prescriptions of geometric quantization, subsequently enables us to give an explicit formula for the operators corresponding to these symmetries. Regarding the first point, i.e. the dimension of the Hilbert space, the story is a bit more involved, and we will now spend a few lines describing the motivation and the results.

If you apply holomorphic quantization to a compact Kähler $2n$-manifold, it is a consequence of the Hirzebruch-Riemann-Roch formula that the dimension of the Hilbert space is finite and grows asymptotically as $\Omega/(2\pi\hbar)^n$ when $\hbar \to 0$, where Ω is the volume of the manifold. This result is also physically interesting, since it agrees with some predictions of semi-classical statistical mechanics. Trying to see what happens on the non-compact $SL(2,\mathbb{C})$ with our invariant metrics, we were thus led to compute the dimension of the Hilbert space. The results obtained can be summarized as follows.

The Hilbert space $\mathcal{H}_{HQ}$ in our setting is essentially the space of square-integrable holomorphic functions on $SL(2,\mathbb{C})$, where square-integrable means with respect to some metric-dependent measure on $SL(2,\mathbb{C})$. Furthermore all these holomorphic functions can be seen as restrictions of holomorphic functions on $\mathbb{C}^4 \supset SL(2,\mathbb{C})$. Defining the subspace $\mathcal{H}_{poly} \subseteq \mathcal{H}_{HQ}$ of the holomorphic functions which are restrictions of polynomials in $\mathbb{C}^4$, we then find that $\dim\mathcal{H}_{poly} \sim \Omega/(2\pi\hbar)^3$ as $\hbar \to 0$ whenever both members are finite. The exact dimension of $\mathcal{H}_{poly}$, which we also compute, depends on the particular invariant metric one puts on $SL(2,\mathbb{C})$; its asymptotic behaviour however does not. This leads us to conjecture that, as in the compact Kähler case, also for general Stein manifolds (i.e. complex submanifolds of $\mathbb{C}^N$) this asymptotic behaviour of $\dim\mathcal{H}_{poly}$ is "universal" – see the discussion of section 8.

2 The invariant Kähler metrics

We start by considering the action of the group $G := SU(2) \times SU(2)$ on the complex manifold $M := SL(2,\mathbb{C})$ defined by

$$\psi : G \times M \longrightarrow M \quad , \quad (U_1, U_2, A) \mapsto U_1 A U_2^{-1} \quad . \tag{1}$$

This is clearly a smooth action which acts on M through biholomorphisms. A detailed study of ψ and its orbits is done in Appendix A. For example one finds there that all the orbits except one have real dimension 5, the exceptional one being $SU(2) \subset M$, which has dimension 3. For the purposes of this section, however, it is enough to quote the following result :

Proposition 2.1. *Any smooth G-invariant function $\tilde{f} : M \to \mathbb{R}$ can be written as a composition $f \circ y$, where $y : M \to [0, +\infty[$ is defined by $y(A) = \cosh^{-1}[\frac{1}{2}\mathrm{tr}(A^\dagger A)]$, and $f : \mathbb{R} \to \mathbb{R}$ is a smooth even function.*

We are now interested in studying Kähler metrics and forms over M. To begin with, the well-known diffeomorphism $M \simeq S^3 \times \mathbb{R}^3$ implies that the de Rham cohomology of M and S^3 are the same. In particular every closed 2-form on M is exact. On the other hand, regarding $\mathbb{C}^4$ as the set of 2×2 complex matrices, we have that M is the hypersurface given as the zero set of the polynomial $A \mapsto 1 - \det A$. Since the derivative of this polynomial is injective on the zero set, M is a complex submanifold of $\mathbb{C}^4$. It then follows from standard results in complex analysis of several variables

(see th. 5.1.5, 5.2.10 and 5.2.6 of [6]) that M is a Stein manifold with Dolbeault groups $H^{p,q}(M) = 0$ (except for $p = q = 0$).

From all this we get the following lemma:

Lemma 2.2. *Any closed (1,1)-form ω on M can be written $\omega = \frac{i}{2}\partial\bar{\partial}\tilde{f}$, where $\tilde{f}$ is a smooth function on M. If ω is real, then $\tilde{f}$ can also be chosen real.*

Proof. This is just like the usual proof of the local $\partial\bar{\partial}$-lemma. As argued above, the closedness of ω implies its exactness, hence $\omega = \mathrm{d}\psi = \partial\psi^{0,1} + \bar{\partial}\psi^{1,0}$ for some $\psi \in H^1(M, \mathbb{C})$. Since $\partial\psi^{0,1} = \bar{\partial}\psi^{1,0} = 0$ (because ω is a (1,1)-form) and $H^{1,0}(M) = H^{0,1}(M) = 0$, we have that $\psi^{0,1} = \bar{\partial}f_1$ and $\psi^{1,0} = \partial f_2$ for some smooth functions f_j on M. Defining $\tilde{f} = 2i(f_2 - f_1)$ we thus get $\omega = \frac{i}{2}\partial\bar{\partial}\tilde{f}$. If $\omega = \frac{i}{2}\partial\bar{\partial}\tilde{f}$ is real, then $\frac{1}{2}(\tilde{f} + \text{c.c.})$ is a real potential for ω. $\square$

Having done this preparatory work, we now head on to the main result of this section.

Proposition 2.3. *Suppose $\omega \in \Omega^{1,1}(M; \mathbb{R})$ is a closed G-invariant form. Then one can always write $\omega = \frac{i}{2}\partial\bar{\partial}(f \circ y)$, where f and y are as in proposition 2.1 and $f \circ y$ is smooth. The function f is unique up to a constant. Furthermore, the hermitian metric on M associated with ω is positive-definite iff $f' > 0$ on $]0, +\infty[$ and $f'' > 0$ on $[0, +\infty[$.*

Proof. By the previous lemma $\omega = \frac{i}{2}\partial\bar{\partial}\tilde{f}$ for some $\tilde{f} \in C^\infty(M; \mathbb{R})$. Now, for any $g \in G$, the G-invariance of ω and the holomorphy of ψ_g imply that

$$\omega = \psi_g^*\omega = \psi_g^* \frac{i}{2}\partial\bar{\partial}\tilde{f} = \frac{i}{2}\partial\bar{\partial}(\tilde{f} \circ \psi_g) \ .$$

Hence by averaging over $g \in G$ if necessary (recall that G is compact), one may assume that the potential $\tilde{f}$ is G-invariant. The first part of the result then follows from proposition 2.1.

To establish the second part, recall that the associated hermitian metric is defined by

$$H(\cdot, \cdot) = \omega(\cdot, J\cdot) - i\,\omega(\cdot, \cdot) \ , \tag{2}$$

where J is the complex structure on M. Since both ω and J are G-invariant (the last one because ψ_g is holomorphic), we conclude that also H is G-invariant. Now consider the complex submanifold $\Lambda \subset M$ consisting of the diagonal matrices in M. It follows from lemma A.1 of Appendix A that Λ intersects every orbit of ψ. Hence, by the G-invariance, H is positive-definite on M iff it is positive-definite at every point of Λ. To obtain the condition for positiveness over Λ we now use a direct computation.

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Take the neighbourhood $\mathcal{U} := \{A \in M : A_{11} \neq 0\}$ and the complex chart ε of M defined by

$$\varepsilon : \mathcal{U} \to \mathbb{C}^* \times \mathbb{C}^2 \quad , \quad \varepsilon^{-1}(z_1, z_2, z_3) = \begin{bmatrix} z_1 & z_2 \\ z_3 & \frac{1+z_2 z_3}{z_1} \end{bmatrix} \tag{3}$$

Note that $\Lambda \subset \mathcal{U}$ and that ε is a chart of M adapted to Λ. Defining $x(A) = \operatorname{tr}(A^\dagger A)/2$ we have that $y = \cosh^{-1}(x)$ and

$$x \circ \varepsilon^{-1}(z) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |1 + z_2 z_3|^2/|z_1|^2\right) .$$

A direct calculation using the chain rule now shows that, on a point $\operatorname{diag}(z_1, z_1^{-1}) \in \Lambda$, we have

$$\omega = \frac{i}{2} \partial \bar{\partial}(f \circ y) = \frac{i}{2}\left[\frac{f''(y)}{|z_1|^2} \, dz_1 \wedge d\bar{z}_1 + \frac{f'(y)}{2 \sinh(y)}\left(dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3\right)\right] \tag{4}$$

and hence

$$H = \frac{f''(y)}{|z_1|^2} \, dz_1 \otimes d\bar{z}_1 + \frac{f'(y)}{2 \sinh(y)}\left(dz_2 \otimes d\bar{z}_2 + dz_3 \otimes d\bar{z}_3\right) . \tag{5}$$

Thus at points of Λ such that $y > 0$ (i.e. $|z_1| \neq 1$), we have $\sinh(y) > 0$ and the positive-definiteness of H is equivalent to $f'(y), f''(y) > 0$. On the other hand, since H and the chart are defined over all of Λ, continuity implies that at a point of Λ with $y = 0$ (i.e. $|z_1| = 1$) we must have

$$H = f''(0) \, dz_1 \otimes d\bar{z}_1 + \frac{1}{2} f''(0) \left(dz_2 \otimes d\bar{z}_2 + dz_3 \otimes d\bar{z}_3\right) ,$$

where it was used that

$$\lim_{y \to 0^+} \frac{f'(y)}{\sinh(y)} = \lim_{y \to 0^+} \frac{f'(y)}{y} = f''(0) .$$

Thus at this point the positive-definiteness of H is equivalent to $f''(0) > 0$. This establishes the last part of the proposition.

To end the proof we finally note that formula (5) implies the uniqueness of $f'(y)$, and hence the uniqueness of f up to a constant. $\quad\square$

Roughly speaking, this proposition guarantees the existence of G-invariant potentials for G-invariant Kähler forms. A particular feature of these potentials, which will be crucial for the explicit calculations later on, is that they are entirely determined by their values on the diagonal matrices, since every orbit of the G-action contains one of these. Having this in mind, we now end this section by presenting a technical lemma which will prove useful on several occasions.

Lemma 2.4. *Suppose $\tilde{f}$ is a smooth G-invariant function on M, and consider the submanifold $\Lambda = \{\mathrm{diag}(z_1, z_1^{-1}) : z_1 \in \mathbb{C}^*\}$ of diagonal matrices in M. If $h = h(|z_1|)$ is a smooth function on Λ such that $\partial\bar{\partial}h = \partial\bar{\partial}\tilde{f}|_\Lambda$, then $2\tilde{f}(z_1) = h(z_1) + h(z_1^{-1}) + \mathrm{const.}$ on the submanifold Λ.*

Proof. The hypothesis is that $\frac{\partial^2 h}{\partial z_1 \partial \bar{z}_1} = \frac{\partial^2 \tilde{f}}{\partial z_1 \partial \bar{z}_1}$ on Λ. Writing $z_1 \in \mathbb{C}^*$ as $z_1 = re^{i\theta}$ and using the expression for the laplacian in polar coordinates, we have

$$0 = \frac{\partial^2(\tilde{f} - h)}{\partial z_1 \partial \bar{z}_1} = \frac{1}{4}\Delta(\tilde{f} - h) = \frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\tilde{f} - h) \;.$$

But the G-invariance implies that $\tilde{f}$ only depends on r; since the same is assumed for h, we get

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\tilde{f} - h) = 0 \quad \Rightarrow \quad \tilde{f} - h = A\log r + B \;.$$

Now, G-invariance also implies that $\tilde{f}(z_1) = \tilde{f}(z_1^{-1})$, thus

$$\begin{aligned}
2\tilde{f}(z_1) &= \tilde{f}(z_1) + \tilde{f}(z_1^{-1}) = h(z_1) + h(z_1^{-1}) + A(\log|z_1| + \log|z_1|^{-1}) + 2B = \\
&= h(z_1) + h(z_1^{-1}) + 2B \;.
\end{aligned}$$

$\square$

3 Curvature and completeness

Throughout this section ω will be the Kähler form of a G-invariant Kähler metric on M. Thus according to proposition 2.3 we can write

$$\omega = \frac{i}{2}\partial\bar{\partial}(f \circ y) \;, \tag{6}$$

where $f \circ y$ is smooth and f satisfies all the conditions of proposition 2.3.

The first task now is to calculate the Ricci form ρ associated to this Kähler metric. More precisely, we will obtain a potential for ρ expressed in terms of the function f.

Proposition 3.1. *The Ricci form of the metric with Kähler form ω is given by*

$$\rho = -i\,\partial\bar{\partial}\log\left[\left(\frac{f'(y)}{\sinh(y)}\right)^2 f''(y)\right] \;.$$

Proof. The G-invariance of the metric implies the G-invariance of the Ricci form ρ. Thus, by proposition 2.3, ρ has a global G-invariant potential $\tilde{\rho}$. Now consider the chart $(\mathcal{U}, z_1, z_2, z_3)$ for M defined in the proof of the same proposition. According to a standard result, if in this chart

$$\omega|_\mathcal{U} = \frac{i}{2}h_{\alpha\bar{\beta}}\,dz^\alpha \wedge d\bar{z}^\beta \;,$$

then the Ricci form is given by

$$\rho|_{\mathcal{U}} \ = \ -i\partial\bar{\partial}\log(\det h_{\alpha\bar{\beta}}) \ .$$

In particular, over the complex submanifold Λ of diagonal matrices we have

$$\frac{i}{2}\partial\bar{\partial}\tilde{\rho}|_{\Lambda} \ = \ \rho|_{\Lambda} \ = \ -i\partial\bar{\partial}\log(\det h_{\alpha\bar{\beta}})|_{\Lambda} \ .$$

But (5) gives us $h_{\alpha\bar{\beta}}$ over Λ, and so we compute that

$$\log(\det h_{\alpha\bar{\beta}})|_{\Lambda} \ = \ \log\left(\frac{1}{|z_1|^2}\left(\frac{f'(y)}{2\sinh y}\right)^2 f''(y)\right) \ .$$

Since this function only depends on $|z_1|$, by lemma 2.4 we get that

$$\tilde{\rho}|_{\Lambda} \ = \ -2\log\left(\left(\frac{f'(y)}{\sinh y}\right)^2 f''(y)\right) + \text{const.} \ .$$

Finally the G-invariance of $\tilde{\rho}$ guarantees that this expression is valid all over M. Thus we conclude that $\rho = \frac{i}{2}\partial\bar{\partial}\tilde{\rho}$ has the stated form. ☐

The next step is the computation of the scalar curvature. Note that the G-invariance of the metric implies the G-invariance of this function.

Proposition 3.2. *The scalar curvature of the Riemannian metric associated with the Kähler form ω is*

$$s \ = \ \frac{2}{f''(f')^2}\frac{\mathrm{d}}{\mathrm{d}y}\left((f')^2\frac{\mathrm{d}}{\mathrm{d}y}\log\left(\frac{\sinh^2 y}{f''(f')^2}\right)\right) \ .$$

Proof. Let us call $g(y) := \log\left(\frac{\sinh^2 y}{f''(f')^2}\right)$, so that $\rho = i\partial\bar{\partial}(g\circ y)$. The same calculations that led to formula (4) now give

$$\rho|_{\Lambda} \ = \ i\left(\frac{g''}{|z_1|^2}\,\mathrm{d}z_1\wedge\mathrm{d}\bar{z}_1 + \frac{g'}{2\sinh y}(\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2 + \mathrm{d}z_3\wedge\mathrm{d}\bar{z}_3)\right) \ . \tag{7}$$

Writing $\omega = \frac{i}{2}h_{\alpha\bar{\beta}}\,\mathrm{d}z^{\alpha}\wedge\mathrm{d}\bar{z}^{\beta}$ and $\rho = \frac{i}{2}r_{\alpha\bar{\beta}}\,\mathrm{d}z^{\alpha}\wedge\mathrm{d}\bar{z}^{\beta}$, the scalar curvature of the associated Riemannian metric is $s = 2h^{\alpha\bar{\beta}}r_{\alpha\bar{\beta}}$. Thus using (4) and (7) we can compute the restriction of s to the submanifold Λ:

$$s|_{\Lambda} \ = \ 2\frac{g''}{f''} + 4\frac{g'}{f'} \ = \ \frac{2}{f''(f')^2}\frac{\mathrm{d}}{\mathrm{d}y}((f')^2 g') \ . \tag{8}$$

The G-invariance of s then shows that this formula is valid all over M. ☐

In the last part of this section we will make contact with a paper by Patrizio and Wong [9]: this will give us almost for free some results about the completeness of the G-invariant metric associated to ω.

To make contact one just needs to note that the linear transformation on $\mathbb{C}^4$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -1 & -i & 0 \\ 1 & 0 & 0 & i \end{bmatrix}$$

takes the standard hyperquadric $Q_4 = \{w \in \mathbb{C}^4 : \sum w_k^2 = 1\}$ to M, and the norm function $\|w\|^2$ on Q_4 to the function $x(A) = \operatorname{tr}(A^\dagger A)/2$ on M. Therefore all the results in [9] valid for $(Q_n, \|w\|^2)$ can be restated here for (M, x). In particular we have that

(1) The function $y = \cosh^{-1} x$ is plurisubharmonic exhaustion on M, and solves the homogeneous Monge-Ampère equation on $M - y^{-1}(0) = M - SU(2)$ ([9], th. 1.2).

(2) Suppose $\tilde{f} = f \circ y$ is a strictly plurisubharmonic function on M. Then with respect to the metric defined by $\frac{i}{2}\partial\bar{\partial}\tilde{f}$, the distance in M between the level sets $\{y = a\}$ and $\{y = b \geq a\}$ is ([9], th. 3.3)

$$D(a,b) = \frac{1}{\sqrt{2}} \int_{f(a)}^{f(b)} \sqrt{-\frac{(f^{-1})''(t)}{(f^{-1})'(t)}}\, dt = \frac{1}{\sqrt{2}} \int_a^b \sqrt{f''(y)}\, dy \; . \tag{9}$$

Furthermore, the distance-minimizing geodesics between these level sets are the integral curves of the vector field $X/\|X\|$, where X is the gradient vector field of $\tilde{f}$ (one can check directly that $X \neq 0$ on $M - SU(2)$).

As a consistency check, we remark that the strict plurisubharmonicity of $\tilde{f} = f \circ y$ together with proposition 2.3 garantees that $f''(y) > 0$ on $[0, +\infty[= y(M)$; thus the integral formula for the distance is well defined. It is now more or less straightforward to prove the following proposition.

Proposition 3.3. *The metric on M with Kähler form ω is complete if and only if*

$$D(0, +\infty) = \frac{1}{\sqrt{2}} \int_0^{+\infty} \sqrt{f''(y)}\, dy = +\infty \; .$$

Proof. By Hopf-Rinow, the metric is complete iff the closed bounded sets of (M, ω) are compact. So suppose that $D(0, +\infty) = +\infty$ and that B is a closed and bounded subset of M. Then for b big enough we have

$$D(0, b) > \sup_{x \in B} D(0, y(x)) \quad \Rightarrow \quad B \subset y^{-1}([0, b]) = x^{-1}([1, \cosh b]) \; .$$

But x is just the usual norm on $\mathbb{C}^4$ restricted to M, thus B is also closed and bounded in $\mathbb{C}^4$, and so is compact.

Conversely, if $D(0, +\infty) < +\infty$, then M itself is a closed bounded set which is not compact, and thus the metric is incomplete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Volume and integration

The purpose of this section is to study the integrals over (M, ω) of G-invariant functions, where ω is as in (6). More precisely, we want to prove the following result.

Proposition 4.1. *Let $\tilde{h}$ be a smooth G-invariant function on M, which by proposition 2.1 can be written $\tilde{h} = h \circ y$, and let M_r be the open submanifold $y^{-1}([0, r[) \subset M$. Then we have that*

$$\int_{M_r} \tilde{h}\, \frac{\omega^3}{3!} \;=\; \frac{\pi^3}{3} \int_0^r h(y)\, \frac{\mathrm{d}}{\mathrm{d}y}(f'(y))^3\, \mathrm{d}y \;. \tag{10}$$

Notice that $\omega^3/3!$ is the volume form of the metric on M associated with ω, so with the particular choice $\tilde{h} \equiv 1$ we get the volume of M_r. Remark also that with $\tilde{h} \equiv 1$ or $\tilde{h} \equiv s$, where s is the scalar curvature given by proposition 3.2, the integral on the right-hand side is trivially computable. Thus taking into account the restrictions on f imposed by propositions 2.1 and 2.3, one gets the following corollary.

Corollary 4.2. *For the Kähler metric on M associated with ω, the volume of M_r and the integral of the scalar curvature over M_r are, respectively,*

$$\frac{1}{3}(\pi f'(r))^3 \qquad \text{and} \qquad 2\pi^3 \left(f'(y)^2 \frac{\mathrm{d}}{\mathrm{d}y} \log \left(\frac{\sinh^2 y}{f''(y) f'(y)^2} \right) \right)_{y=r} \;.$$

In particular M has finite volume iff $f'(r)$ is bounded.

We now embark on the proof of proposition 4.1. To start with, it will be convenient to restate here some results used in [10, 11] to study the lump metric.

Consider the Pauli matrices

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;,\quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \;,\quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \;,$$

so that $\{\frac{i}{2}\tau_a\}$ is a basis for the Lie algebra $su(2)$. Associated to each $\frac{i}{2}\tau_a$ is a left-invariant 1-form σ_a on $SU(2)$, and $\{\sigma_a\}$ is a global trivialization of the cotangent bundle of $SU(2)$. Then according to [10, 11] and the references therein we have that :

- There is a diffeomorphism $\chi : SU(2) \times \mathbb{R}^3 \to M$ defined by

$$\chi(U, \vec{\lambda}) = U \left(\sqrt{1 + \lambda^2} I + \vec{\lambda} \cdot \vec{\tau} \right) \;, \quad \text{with} \quad \lambda = |\vec{\lambda}| \;.$$

- The usual action ψ of G on M is taken by χ to the action $\tilde{\psi}$ on $SU(2) \times \mathbb{R}^3$ given by

$$\tilde{\psi}_{(U_1,U_2)}(U, \vec{\lambda}) \;=\; (U_1 U U_2^{-1}, \; \mathcal{R}_{U_2}(\vec{\lambda})) \qquad (11)$$

where $\mathcal{R} : SU(2) \to SO(3)$ is the usual double covering; explicitly $\mathcal{R}_{U_2} \in SO(3)$ has components $(\mathcal{R}_{U_2})_{ab} = \frac{1}{2}\mathrm{tr}(\tau_a U_2 \tau_b U_2^\dagger)$.

- Regarding the σ_a and the $\mathrm{d}\lambda_a$ as 1-forms defined over $SU(2) \times \mathbb{R}^3$, the action $\tilde{\psi}$ acts on these forms by $\tilde{\psi}^*_{(U_1,U_2)}(\vec{\sigma}, \mathrm{d}\vec{\lambda}) = (\mathcal{R}_{U_2}\vec{\sigma}, \mathcal{R}_{U_2}\mathrm{d}\vec{\lambda})$.

- The Euler angles $(\beta, \alpha, \gamma) \in]0, 4\pi[\times]0, \pi[\times]0, 2\pi[$ define an oriented chart of $SU(2)$ with dense domain such that, on this domain,

$$
\begin{aligned}
\sigma_1 &= -\sin\gamma \; \mathrm{d}\alpha + \cos\gamma \sin\alpha \; \mathrm{d}\beta \\
\sigma_2 &= \cos\gamma \; \mathrm{d}\alpha + \sin\gamma \sin\alpha \; \mathrm{d}\beta \qquad (12) \\
\sigma_3 &= \cos\alpha \; \mathrm{d}\beta + \mathrm{d}\gamma \; .
\end{aligned}
$$

The plan now is to use the diffeomorphism χ to compute the integrals on $SU(2)\times\mathbb{R}^3$, instead of M. Since the $\{\sigma_a, \lambda_a\}$ trivialize the cotangent bundle of $SU(2) \times \mathbb{R}^3$, the pull-back by χ of the volume form on M can be written

$$\mu \;:=\; \chi^* \frac{\omega^3}{3!} \;=\; \hat{\mu}(U, \vec{\lambda}) \; \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3 \; ,$$

for some non-vanishing function $\hat{\mu}$ on $SU(2)\times\mathbb{R}^3$. Moreover, μ must be invariant under $\tilde{\psi}$, because the volume form on M is invariant under ψ. But notice now that, under $\tilde{\psi}$,

$$\vec{\sigma} \mapsto \mathcal{R}_{U_2}\vec{\sigma} \quad \Rightarrow \quad \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \mapsto \det(\mathcal{R}_{U_2})\, \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \;=\; \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \; ,$$

because $\mathcal{R}_{U_2} \in SO(3)$. For the same reason, also $\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3$ is invariant, and hence $\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3$ is invariant too. This fact together with the invariance of μ implies the invariance of the function $\hat{\mu}$. From the formula (11) for the action $\tilde{\psi}$ it is then clear that $\hat{\mu}$ only depends on $\lambda = |\vec{\lambda}|$.

The computation of the function $\hat{\mu}(\lambda)$ is now straightforward. First we have

$$
\begin{aligned}
\hat{\mu}(\lambda) &= \mu_{(\mathrm{Id},0,0,\lambda)} \left(\frac{i}{2}\tau_1, \frac{i}{2}\tau_2, \frac{i}{2}\tau_3, \frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \frac{\partial}{\partial\lambda_3} \right) = \\
&= \frac{1}{6}\, (\omega^3)_{\chi(\mathrm{Id},0,0,\lambda)} \left(\chi_*(\frac{i}{2}\tau_1), \; \ldots, \; \chi_* \frac{\partial}{\partial\lambda_3} \right) \; .
\end{aligned}
$$

On the other hand, using the chart (3) and (4), at the point $q(\lambda) := \chi(\mathrm{Id}, 0, 0, \lambda) = \mathrm{diag}(\sqrt{1+\lambda^2} + \lambda, \sqrt{1+\lambda^2} - \lambda)$ of M we also have

$$\frac{1}{6}\, (\omega^3)_{q(\lambda)} \;=\; (\frac{i}{2})^3 \frac{f''(y)}{(\sqrt{1+\lambda^2}+\lambda)^2} \left(\frac{f'(y)}{2\sinh y} \right)^2 \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}\bar{z}_3 \; .$$

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Finally a tedious calculation that we will not reproduce shows that

$$(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}\bar{z}_3)_{q(\lambda)} \left(\chi_*(\frac{i}{2}\tau_1), \ldots, \chi_* \frac{\partial}{\partial \lambda_3} \right) = 4i\sqrt{1+\lambda^2}(\sqrt{1+\lambda^2}+\lambda)^2 \,,$$

and so we get

$$\hat{\mu}(U, \vec{\lambda}) = \hat{\mu}(\lambda) = \frac{\sqrt{1+\lambda^2}}{8} \left(\frac{f'(y \circ \chi)}{\sinh(y \circ \chi)} \right)^2 f''(y \circ \chi) \,.$$

Having calculated the volume form on $SU(2) \times \mathbb{R}^3$, the rest of the proof of proposition 4.1 goes on smoothly.

Call as usual $x(A) = \mathrm{tr}(A^\dagger A)/2$ and $y = \cosh^{-1}(x)$. A quick calculation shows that $x \circ \chi(U, \vec{\lambda}) = 1+2\lambda^2$, and so we have an explicit relation $y = y(\lambda)$. From this relation it is clear that $\chi^{-1}(M_r) = SU(2) \times B_l$, where B_l is the open ball, centered at the origin of $\mathbb{R}^3$, with radius l such that $1 + 2l^2 = \cosh r$. Hence, for any invariant function $\tilde{h} = h \circ y$ on M we have

$$\int_{M_r} \tilde{h} \frac{\omega^3}{3!} = \int_{\chi^{-1}(M_r)} (\tilde{h} \circ \chi) \mu = \int_{SU(2) \times B_l} (h \cdot \hat{\mu})(y(\lambda)) \, \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3$$

$$= \left(\int_{SU(2)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \right) \int_0^l (h \cdot \hat{\mu})(y(\lambda)) \, 4\pi\lambda^2 \, \mathrm{d}\lambda \,.$$

Using the value of $\hat{\mu}(\lambda)$ and the relation $y = y(\lambda)$, a change of variables in the last integral shows that it coincides with

$$\frac{\pi}{16} \int_0^r h(y) \, f''(y) \, (f'(y))^2 \, \mathrm{d}y \,.$$

The first integral can be computed using (12). Namely we have

$$\int_{SU(2)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \int_0^{2\pi} \int_0^{\pi} \int_0^{4\pi} \sin\alpha \, \mathrm{d}\beta \, \mathrm{d}\alpha \, \mathrm{d}\gamma = 16\pi^2 \,.$$

Putting these two results together we finally obtain the formula stated in proposition 4.1.

5   Examples

5.1   The one-lump metric

The so-called moduli space of degree 1 lumps on a sphere, which we will call $\mathcal{M}$, is just the group of rational maps $S^2 \to S^2$. Identifying $S^2 \simeq \mathbb{C}P^1$, this group is the same as the group of projective transformations

$$PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^* = SL(2, \mathbb{C})/\{\pm 1\} \,.$$

In the physics literature, $\mathcal{M}$ is the space of minimal energy static solutions of the sigma-model defined on the Lorentzian spacetime $S^2 \times \mathbb{R}$ with S^2 as target space. The kinetic energy functional of this sigma-model induces a certain Riemannian metric on $\mathcal{M}$, which is also very natural geometrically. It can be defined in the following way.

Let $w_t : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a one parameter family of projective transformations, i.e. a curve on $\mathcal{M}$, and call w_0' its tangent vector at $t = 0$. For each $x \in \mathbb{C}P^1$, $t \mapsto w_t(x)$ is a curve in $\mathbb{C}P^1$, and we call $v(x) \in T_{w_0(x)}\mathbb{C}P^1$ its tangent vector at $t = 0$. Then the Riemannian metric g on $\mathcal{M}$ is defined by

$$g(w_0', w_0') := \int_{x \in \mathbb{C}P^1} h(v(x), v(x)) \operatorname{vol}_h \tag{13}$$

where h is the Fubini-Study metric on $\mathbb{C}P^1$ and vol_h is the associated volume form. In informal terms, one may say that the squared-length of an infinitesimal curve $t \mapsto w_t$ in $(\mathcal{M}, g)$ is just the average over $x \in \mathbb{C}P^1$ of the squared-lengths of the infinitesimal curves $t \mapsto w_t(x)$ in $(\mathbb{C}P^1, h)$; thus the measure of "displacement" in $\mathcal{M}$ is how much the image points of w_t are moved. Using the fact that transformations in $PSU(2) \subset PGL(2, \mathbb{C})$ are isometries of $(\mathbb{C}P^1, h)$, it is not difficult to check that right and left multiplication in $PGL(2, \mathbb{C})$ by elements of $PSU(2)$ are in fact isometries of $(\mathcal{M}, g)$.

Now consider the usual chart of the projective space $\mathbb{C}P^1 \setminus \{[0, 1]\} \to \mathbb{C}$, $[1, z] \mapsto z$, and let (u^1, u^2, u^3) be any complex chart of $\mathcal{M}$ defined on a neighbourhood of the point w_0. In these charts we have

$$w_0' = \frac{\mathrm{d}u^j}{\mathrm{d}t}(0) \frac{\partial}{\partial u^j}$$

$$v(z) = \frac{\mathrm{d}}{\mathrm{d}t} w_t(z) = \frac{\mathrm{d}}{\mathrm{d}t} w_{u(t)}(z) = \frac{\partial}{\partial u^j}(w_u(z)) \frac{\mathrm{d}u^j}{\mathrm{d}t}(0) \frac{\partial}{\partial z}$$

$$h(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = h_{1\bar{1}} = \frac{\partial^2}{\partial z \partial \bar{z}} \log(1 + |z|^2)$$

$$\operatorname{vol}_h = \frac{i}{2} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2}$$

where the last two equalities are standard properties of the Fubini-Study metric. Calling $\rho := \log(1 + |z|^2)$ the local potential of the Fubini-Study metric we get

$$g(w_0', w_0') = \int_{z \in \mathbb{C}} \frac{\partial^2 \rho}{\partial z \partial \bar{z}}(w_0(z)) \frac{\partial(w_u(z))}{\partial u^j} \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\partial(\bar{w}_u(z))}{\partial \bar{u}^k} \frac{\mathrm{d}\bar{u}^k}{\mathrm{d}t} \frac{i}{2} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2} =$$

$$= \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}\bar{u}^k}{\mathrm{d}t} \int_{z \in \mathbb{C}} \frac{\partial^2}{\partial u^j \partial \bar{u}^k}[\rho(w_u(z))] \frac{i}{2} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2} =$$

$$= \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}\bar{u}^k}{\mathrm{d}t} \frac{\partial^2}{\partial u^j \partial \bar{u}^k} \frac{i}{2} \int_{z \in \mathbb{C}} \rho[w_u(z)] \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2} \ .$$

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Since this equation is valid in any chart (u^k) of $\mathcal{M}$, we conclude that the function

$$a(w) := \frac{i}{2} \int_{z \in \mathbb{C}} \log(1 + |w(z)|^2) \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2}$$

is a global Kähler potential for the Kähler form on $\mathcal{M}$ associated with the Riemannian metric g. Calling this form ω, we thus have $\omega = \frac{i}{2} \partial \bar{\partial} a$.

It turns out, however, that the integral defining $a(w)$ is difficult to compute for a general $w \in PGL(2, \mathbb{C})$, and so we cannot calculate the potential directly. To circumvent this obstacle we proceed in the following way.

Firstly, using the double cover $\pi : SL(2, \mathbb{C}) \to PGL(2, \mathbb{C})$, we work on the more palpable group $SL(2, \mathbb{C})$. Notice that $\pi^* \omega = \frac{i}{2} \partial \bar{\partial}(a \circ \pi)$, because π is holomorphic. Moreover, the invariance of g and ω by right and left multiplication by elements of $PSU(2)$, implies that $\pi^* \omega$ is invariant by the usual action ψ of the group G on $SL(2, \mathbb{C})$. Thus we are on familiar ground. From proposition 2.3 we get that $\pi^* \omega = \frac{i}{2} \partial \bar{\partial} \tilde{f}$, for some G-invariant function $\tilde{f}$. The plan now is to compute $\tilde{f}$ using the potential $a(w)$ and lemma 2.4.

In fact, for a diagonal matrix $A = \mathrm{diag}(\xi, \xi^{-1})$ one can compute that

$$a \circ \pi(A) = \frac{i}{2} \int_{z \in \mathbb{C}} \log\left(1 + \left|\frac{z}{\xi^2}\right|^2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2} =$$
$$= 2\pi \int_0^{+\infty} \log\left(1 + \frac{r^2}{|\xi|^4}\right) \frac{r}{(1 + r^2)^2} \, \mathrm{d}r = \pi \frac{\log |\xi|^4}{|\xi|^4 - 1},$$

and since

$$\partial \bar{\partial}(a \circ \pi)|_\Lambda = -2i(\pi^* \omega)|_\Lambda = \partial \bar{\partial} \tilde{f}|_\Lambda,$$

from lemma 2.4 we get that

$$2\tilde{f}|_\Lambda = 2\pi \frac{|\xi|^4 + 1}{|\xi|^4 - 1} \log |\xi|^2.$$

Now using the formulas $x(A) = \mathrm{tr}(A^\dagger A)/2 = (|\xi|^2 + |\xi|^{-2})/2$ and $y = \cosh^{-1}(x)$, a little algebra shows that, over $\Lambda \subset SL(2, \mathbb{C})$,

$$\tilde{f} = \pi \frac{x}{\sqrt{x^2 - 1}} \log(x + \sqrt{x^2 - 1}) = \pi y \coth y. \tag{14}$$

The G-invariance of $\tilde{f}$ finally guarantees that this formula is valid all over $SL(2, \mathbb{C})$. We have thus obtained an explicit potential for the Kähler form $\pi^* \omega$. Notice that $\tilde{f}(A) = \tilde{f}(-A)$ for any matrix in $SL(2, \mathbb{C})$, and so $\tilde{f}$ descends to a function on $PGL(2, \mathbb{C})$; this will be a potential for the Kähler form ω on this space.

Using the potential function $\tilde{f}$ and the results of the previous sections, we will now derive a series of properties of the metric g. Except for the volume and the Ricci

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potential computations, these properties were already obtained in [11], using different methods.

Substituting expression (14) into propositions 3.1 and 3.2, we obtain a potential for the Ricci form and the scalar curvature in $(M, \pi^*\omega)$. The first is

$$\tilde{\rho}(y) = -2\log(y\cosh y - \sinh y)(\sinh 2y - 2y)^2/(\sinh y)^9$$

and the second has a rather long expression which we will not transcribe. The plot of this expression, however, coincides with the one in [11] [1], and thus the scalar curvature is a positive increasing function of y that diverges at infinity. It is worthwhile noting that, for this metric, the positiveness of the scalar curvature actually comes from the positive-definiteness of the Ricci tensor, as can be seen by applying proposition 2.3 to the potential $\tilde{\rho}$. Using the criterion of proposition 3.3 one may also easily verify that the metric g is incomplete. Finally, from corollary 4.2, and introducing a factor $1/2$ to account for the double cover $(M, \pi^*g) \to (\mathcal{M}, g)$, we obtain that the volume of the moduli space is

$$\mathrm{vol}(\mathcal{M},\, g) \;=\; \frac{\pi^6}{6} \; .$$

5.2 Other metrics

We will briefly mention here other examples of G-invariant metrics on M; these are interesting for their curvature properties.

First of all it is clear from proposition 3.1 that any solution of

$$\frac{\mathrm{d}}{\mathrm{d}y}(f'(y))^3 \;=\; c\,(\sinh y)^2 \quad , \quad c > 0 \; ,$$

will give rise to a Ricci-flat metric on M. This metric coincides with the Stenzel metric on $TS^3 \simeq SL(2, \mathbb{C})$ [12], as can be seen by using the correspondence $\mathcal{M} \leftrightarrow Q_4$ described in section 3 and comparing with section 7 of [12]. It is a complete metric.

Experimenting with other even functions $f(y)$ one can find metrics with a wide range of behaviours. For example it follows from propositions 3.3, 3.1 and 2.3 that the metrics defined by $f(y) = y^2$ and $f(y) = \cosh y$ are complete and have, respectively, positive-definite and negative-definite Ricci tensor. The last one is just the induced metric by the natural inclusion $M \subset \mathbb{C}^4$. The first one has also the pleasant property that the parameter y is precisely the geodesic distance from the submanifold $SU(2) \subset M$, and so the volume of M_r grows exactly with the cube of this distance (see (9) and corollary 4.2).

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[1] Actually our scalar curvature is half of the one in [11], but this must be due to different conventions.

Part II
Holomorphic Quantization

In the second part of the paper we want to study the holomorphic quantization of the Kähler manifolds $(SL(2,\mathbb{C}), \omega)$, where ω is any G-invariant Kähler form. We will firstly obtain the quantum operators corresponding to the classical symmetries of the system. After that we will compute the dimension of the Hilbert space of the quantized system. This last calculation takes a bit of work, but in the end we find some physically satisfactory results, as described in the Introduction.

6   The classical moment map

Recall the action $\psi : G \times M \to M$ described in section 1, and suppose $\omega = \frac{i}{2}\partial\bar{\partial}(f \circ y)$ is any G-invariant Kähler form on M (see proposition 2.3). Then, tautologically, ψ is a symplectic action on (M, ω). Since G is a compact semi-simple Lie group, general results state that there is a unique moment map $\mu : M \to \mathfrak{g}^*$ associated with this action. We will now give an explicit formula for μ.

Proposition 6.1. *For any $m \in M$ and $(a,b) \in \mathfrak{g} = su(2) \oplus su(2)$ we have*

$$\mu(a,b) \;=\; \frac{i}{4}\frac{f'(y)}{\sinh y}\,\mathrm{tr}(mm^\dagger a - m^\dagger m b)$$

where $su(2)$ is identified with the space of 2×2 anti-hermitian matrices, and $y = y(m)$ is the function defined in section 1.

Proof. Since $\omega = -d\alpha$, where $\alpha = \frac{i}{2}\partial(f \circ y)$ is a G-invariant 1-form on M, a well known result (th. 4.2.10 of [1]) states that the moment map satisfies

$$\mu(m)\,[X] \;=\; \alpha_m(X^\#) \tag{15}$$

for any $m \in M$ and $X \in \mathfrak{g}$, where $X^\#$ is the vector field on M generated by the one-parameter group of biholomorphisms $\psi_{\exp(tX)} : M \to M$. Explicitly, for any $(a,b) \in \mathfrak{g} = su(2) \oplus su(2)$ one can compute

$$(a,b)^\#_m \;=\; \frac{\mathrm{d}}{\mathrm{d}t}(e^{ta}me^{-tb})_{t=0} \;=\; am - mb \;, \tag{16}$$

where we regard $T_m M \subset T_m GL(2,\mathbb{C}) \simeq M(2,\mathbb{C})$.

   On the other hand, for each $m \in M \setminus SU(2)$, the formula $\tilde{y} := \cosh^{-1}(\frac{1}{2}\mathrm{tr}A^\dagger A)$ gives a local extension of y to a neighbourhood in $M(2,\mathbb{C})$ of m. Since M is a complex

submanifold of $M(2,\mathbb{C}) \simeq \mathbb{C}^4$, it is then true that $\partial(f \circ \tilde{y})|_{T_m M} = \partial(f \circ y)_m$. Applying these formulas we thus get

$$
\begin{aligned}
\alpha_m[(a,b)^\#] &= \frac{i}{2} f'(y) \sum_{k=1}^4 \frac{\partial \tilde{y}}{\partial z_k} \, \mathrm{d}z_k(am - mb) = \\
&= \frac{i}{2} f'(y) \sum_{k=1}^4 \frac{1}{2 \sinh y} \, \bar{z}_k(m) \, \mathrm{d}z_k(am - mb) = \\
&= \frac{i}{4} \frac{f'(y)}{\sinh y} \sum_{k,l=1}^2 \bar{m}_{kl} \, (am - mb)_{kl} = \frac{i}{4} \frac{f'(y)}{\sinh y} \, \mathrm{tr}(m^\dagger am - m^\dagger mb) \ . \quad (17)
\end{aligned}
$$

Since α and $(a,b)^\#$ are smooth on M, this formula can be extended by continuity from $M \setminus SU(2)$ to M. It coincides with the formula in the statement because of the cyclic property of the trace. $\qquad\square$

Remark. Although we will not reproduce the calculations here, a number of properties of the moment map μ can be obtained quite straightforwardly. For example, with respect to the norm on $su(2)^* \oplus su(2)^*$ induced by the norm $-\mathrm{tr}\, a^2$ on $su(2)$, one has

$$
\|\mu(m)\|^2 = \frac{1}{4} f'(y(m))^2
$$

$$
\mu(M) = \left\{ (a,b) \in su(2)^* \oplus su(2)^* : \ \|a\| = \|b\| \ \in [0, \frac{1}{2\sqrt{2}} f'(+\infty)[\right\}
$$

The moment map obtained above associates to each $X \in \mathfrak{g}$ a function $\mu(\cdot)[X] \in C^\infty(M)$. In the framework of geometric quantization this function is regarded as a classical observable, and the quantization procedure associates to it a certain hermitian operator on the quantum Hilbert space. This correspondence is the subject of the next section.

7 Holomorphic quantization

In this section we want to study the quantization of the classical phase space (M, ω). We will use holomorphic quantization, which is the simplest and most natural quantization procedure on a Kähler manifold. Refinements such as the metaplectic correction will be left out. For background material consult for example [14]. [2]

We start with prequantization. Since the Kähler form $\omega = \frac{i}{2} \partial \bar{\partial}(f \circ y)$ is exact on M, the trivial line-bundle $B := M \times \mathbb{C}$ with the canonical hermitian metric $((m, w_1), (m, w_2)) = w_1 \bar{w}_2$ is a prequantum bundle. Now consider the natural unitary

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[2] A warning about conventions: if (M, ω) is a symplectic manifold and $H \in C^\infty(M)$, the definition of the symplectic gradient vector field X_H used in [14] differs by a sign from ours.

trivialization of this bundle $m \mapsto (m, 1)$, and the connection ∇ on B defined by the 1-form

$$\theta = \frac{1}{4\hbar}(\bar{\partial} - \partial)(f \circ y)$$

with respect to this trivialization. The curvature form of ∇ is $\mathrm{d}\theta = -i\hbar^{-1}\omega$ and, since θ is pure imaginary, the connection is compatible with the hermitian metric $(\cdot, \cdot)$. Thus according to the definitions in [14] $(B, (\cdot, \cdot), \nabla)$ is the prequantum data.

The step from prequantization to quantization is made by choosing a polarization on M. Since M is Kähler the natural choice here is the holomorphic polarization, that is, the polarization spanned by the tangent vectors $\frac{\partial}{\partial z^k}$. With this choice, a section $m \mapsto \varphi(m) = (m, \tilde{\varphi}(m))$ of B is polarized iff $\nabla^{0,1}\varphi = 0$, where $\nabla^{0,1}$ denotes the anti-holomorphic part of the connection. But

$$\nabla^{0,1}\varphi = \bar{\partial}\tilde{\varphi} + \tilde{\varphi}\,\theta^{0,1} = \bar{\partial}\tilde{\varphi} + \frac{1}{4\hbar}\tilde{\varphi}\,\bar{\partial}(f \circ y) = 0 \iff \tilde{\varphi} = \phi\,e^{-(f\circ y)/4\hbar} \qquad (18)$$

where ϕ is any holomorphic function on M. Thus the space of polarized sections of B can be identified with the space of smooth functions on M of the form (18).

The final step to construct the quantum Hilbert space is to define an inner product of polarized sections. This is done by the formula

$$\langle \varphi_1, \varphi_2 \rangle = \int_M (\varphi_1, \varphi_2)\,\epsilon = \int_M \phi_1\bar{\phi}_2 e^{-(f\circ y)/2\hbar}\,\epsilon\,, \qquad (19)$$

where $\epsilon := (2\pi\hbar)^{-3}\omega^3/3!$ differs from the metric volume form on (M, ω) by the factor $(2\pi\hbar)^{-3}$. The quantum Hilbert space of holomorphic quantization, which we denote $\mathcal{H}_{HQ}$, is then defined as the space of polarized sections of B of finite $\langle \cdot, \cdot \rangle$-norm (see [14]).

For a better understanding of this Hilbert space, one should get a clearer picture of the holomorphic functions ϕ on M. This picture is provided by the next proposition. Since its proof is rather out of context and may easily be skipped, we defer the proof to the end of the section.

Proposition 7.1. *Regard $M = SL(2, \mathbb{C})$ as the zero set in $\mathbb{C}^4$ of the polynomial $D(z) = z_1z_4 - z_2z_3 - 1$. Then the natural restriction is an isomorphism between the rings of holomorphic functions $\mathcal{O}(\mathbb{C}^4)/J$ and $\mathcal{O}(M)$, where J is the ideal of $\mathcal{O}(\mathbb{C}^4)$ generated by $D(z)$.*

In other words, this proposition states that every holomorphic function on M is the restriction of a holomorphic function on $\mathbb{C}^4$, and furthermore two holomorphic functions on $\mathbb{C}^4$ restrict to the same function on M iff their difference is divisible by $D(z)$. We thus get a characterization of holomorphic functions on M in terms of entire functions on $\mathbb{C}^4$, which have a global power series expansion and are generally much better understood.

For the rest of this section we will look at operators on $\mathcal{H}_{HQ}$. If $h \in C^\infty(M)$ is a classical observable, geometric quantization associates to it an operator $\hat{h}$ on $\mathcal{H}_{HQ}$ defined by

$$\hat{h}(\varphi) \;=\; i\hbar \, \nabla_{Y_h} \varphi + h \, \varphi \qquad \forall \varphi \in \mathcal{H}_{HQ} \;, \tag{20}$$

where Y_h is the vector field on M defined by $\iota_{Y_h} \omega = \mathrm{d}h$. This observable-operator correspondence does not always work, however, because sometimes the resulting operator $\hat{h}$ takes polarized sections into non-polarized ones. To prevent this, one further demands that the flow of Y_h should preserve the polarization, i.e. the flow should be locally holomorphic. Thus in principle not all observables $h \in C^\infty(M)$ can be "quantized" by this method. It can be shown, however, that if this condition is fullfilled then $\hat{h}$ is a self-adjoint operator in the Hilbert space $(\mathcal{H}_{HQ}, \langle \cdot, \cdot \rangle)$ (see [14] and review [3]).

We will now apply formula (20) to the observables coming from the classical symmetries of (M, ω), that is to the functions $\mu^X := \mu(\cdot) \, [X] \in C^\infty(M)$ described in the previous section. Notice that, by definition of moment map, for each $X \in \mathfrak{g}$ the vector field Y_{μ^X} is exactly $X^\#$ – the vector field generated by the one-parameter group of biholomorphisms $\psi_{\exp(tX)} : M \to M$. In particular the flow of Y_{μ^X}, which is $\psi_{\exp(tX)}$, preserves the holomorphic polarization, and so formula (20) may be applied to μ^X.

Putting together (18), (20) and (15) we get

$$\hat{\mu}^X \, \varphi \;=\; i\hbar \, \nabla_{X^\#} \varphi + \mu^X \varphi \;=\; i\hbar \left[(\mathrm{d}\varphi)(X^\#) + \varphi \, \theta(X^\#) \right] + \alpha(X^\#) \, \varphi \;=$$

$$= \; i\hbar \left[(\mathrm{d}\phi)(X^\#) - \frac{1}{4\hbar} \phi \, \mathrm{d}(f \circ y)(X^\#) + \frac{1}{4\hbar} \phi \, (\bar{\partial} - \partial)(f \circ y) \, (X^\#) \right] e^{-(f \circ y)/4\hbar} +$$

$$+ \; \frac{i}{2} \partial (f \circ y) \, (X^\#) \, \varphi \;=\; i\hbar (\partial \phi)(X^\#) \, e^{-(f \circ y)/4\hbar} \;. \tag{21}$$

For an even more explicit formula, suppose $X = (a, b) \in su(2) \oplus su(2)$ and that $\phi \in \mathcal{O}(M)$ is the restriction of a certain $\tilde{\phi} \in \mathcal{O}(\mathbb{C}^4)$. Then using (16) and the fact that M is a complex submanifold of $\mathbb{C}^4$,

$$(\partial \phi)_m (X^\#) = \sum_{k=1}^{4} \frac{\partial \tilde{\phi}}{\partial z^k}(m) \, z^k(am - mb) \;, \tag{22}$$

where $z^k(am - mb)$ stands for the entry z^k of the matrix $am - mb$ under the identification $M(2, \mathbb{C}) \simeq \mathbb{C}^4$. Formulas (21) and (22) thus give an explicit description of the operator $\hat{\mu}^X$ on $\mathcal{H}_{HQ}$.

Proof of proposition 7.1. This is a known consequence of textbook results. Let $\mathcal{A}_{\mathbb{C}^4}$ and $\mathcal{A}_M$ be the sheaves of germs of holomorphic functions on $\mathbb{C}^4$ and M, respectively. By theorem 7.15 of [6] these are coherent analytic sheaves. Furthermore, calling $\tilde{\mathcal{A}}_M$ the trivial extension to $\mathbb{C}^4$ of the sheaf $\mathcal{A}_M$ over M, it follows from theorems IV-D8 and VI-B5 of [4] that $\tilde{\mathcal{A}}_M$ is still coherent analytic and has the same cohomology as $\mathcal{A}_M$.

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Consider now the short sequence of sheaves over $\mathbb{C}^4$:

$$0 \longrightarrow \mathcal{A}_{\mathbb{C}^4} \xrightarrow{\tilde{D}} \mathcal{A}_{\mathbb{C}^4} \xrightarrow{r} \tilde{\mathcal{A}}_M \longrightarrow 0 \, , \tag{23}$$

where $\tilde{D}$ is the map induced by local multiplication by the polynomial $D(z)$ and r is defined by

$$r|_U(f) \;=\; \begin{cases} 0 \ \text{ if } U \cap M = \emptyset \\ f|_{U \cap M} \ \text{ otherwise} \end{cases} \qquad \in \; \Gamma(U, \tilde{\mathcal{A}}_M)$$

for every open set U in $\mathbb{C}^4$ and every $f \in \Gamma(U, \mathcal{A}_{\mathbb{C}^4})$. It is not difficult to check that (23) is in fact an exact sequence. (Succinctly, $\tilde{D}$ is injective because the stalks of $\mathcal{A}_{\mathbb{C}^4}$ are integral domains; r is surjective because M is a complex submanifold of $\mathbb{C}^4$; $\ker r \subseteq \operatorname{im} \tilde{D}$ by the Nullstellensatz for germs of varieties and the irreducibility of $D(z)$.)

We therefore obtain an exact sequence of cohomology groups

$$0 \longrightarrow H^0(\mathbb{C}^4, \mathcal{A}_{\mathbb{C}^4}) \longrightarrow H^0(\mathbb{C}^4, \mathcal{A}_{\mathbb{C}^4}) \longrightarrow H^0(M, \mathcal{A}_M) \longrightarrow 0 \, ,$$

where we have used that $H^p(\mathbb{C}^4, \tilde{\mathcal{A}}_M) \simeq H^p(M, \mathcal{A}_M)$ and that, by Cartan's theorem B [4, p. 243], $H^1(\mathbb{C}^4, \mathcal{A}_{\mathbb{C}^4}) = 0$. Since the zeroth cohomology groups are just the global sections of the respective sheaf and, under this identification, the first and second maps are, respectively, multiplication by $D(z)$ and the natural restriction, we finally obtain that

$$\mathcal{O}(M) \;=\; \Gamma(M, \mathcal{A}_M) \simeq \frac{\Gamma(\mathbb{C}^4, \mathcal{A}_{\mathbb{C}^4})}{D(z) \cdot \Gamma(\mathbb{C}^4, \mathcal{A}_{\mathbb{C}^4})} \;=\; \mathcal{O}(\mathbb{C}^4)/J \, .$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

8   Dimension of the Hilbert space

In this last section of the paper we will be concerned with the dimension of the quantum Hilbert space associated with the Kähler manifold (M, ω). More specifically, using the identification of the previous section

$$\mathcal{H}_{HQ} \simeq \left\{ \phi \in \mathcal{O}(M) = \mathcal{O}(\mathbb{C}^4)/J : \int_M |\phi|^2 e^{-(f \circ y)/2\hbar} \, \epsilon < +\infty \right\} \, ,$$

let $\mathcal{H}_{poly}$ be the subspace of $\mathcal{H}_{HQ}$ consisting of the holomorphic functions that can be represented by polynomials in $\mathbb{C}^4$. Then we will be able to compute the dimension of $\mathcal{H}_{poly}$ in terms of the Kähler potential f. Furthermore, when (M, ω) has finite volume Ω and $\mathcal{H}_{poly}$ has finite dimension, we will show that $\dim_{\mathbb{C}} \mathcal{H}_{poly} \sim \Omega/(2\pi\hbar)^3$ as $\hbar \to 0^+$. These results are finally discussed in Questions 1, 2 and 3.

The main step towards proving the stated results is the following proposition, which will be proved at the end of this section.

Proposition 8.1. *Let ϕ be a holomorphic function on M that can be represented by a polynomial in $\mathbb{C}^4$, of degree l, whose homogeneous term of highest degree is not divisible by $z_1 z_4 - z_2 z_3$. Then ϕ is in $\mathcal{H}_{poly}$ if and only if*

$$\int_0^{+\infty} (\cosh y)^l \, e^{-f(y)/2\hbar} \, \frac{\mathrm{d}}{\mathrm{d}y} [f'(y)]^3 \, \mathrm{d}y \quad < \quad +\infty \, . \tag{24}$$

Using this proposition the dimension of $\mathcal{H}_{poly}$ can be computed quite straightforwardly. In fact, assuming that $\{l \in \mathbb{N} : (24) \text{ is satisfied}\}$ is not empty (which will be shown to be true when (M, ω) has finite volume), and calling $m \in \mathbb{N} \cup \{+\infty\}$ the maximum of this set, we have:

Corollary 8.2. *The complex dimension of $\mathcal{H}_{poly}$ is $\frac{1}{6}(m+1)(m+2)(2m+3)$.*

Proof. Let $P_l \subset \mathcal{O}(\mathbb{C}^4)$ be the subspace of homogeneous polynomials of degree l, and $P_{\leq m}$ the space $\bigoplus_{0 \leq l \leq m} P_l$. Calling $\chi : \mathcal{O}(\mathbb{C}^4) \to \mathcal{O}(M)$ the natural homomorphism, let also $\chi_|$ be the restriction of χ to $P_{\leq m}$.

By proposition 8.1 we have that $\mathcal{H}_{poly} = \chi(P_{\leq m})$, thus

$$\dim \mathcal{H}_{poly} \;=\; \dim(P_{\leq m}) - \dim(\ker \chi_|) \;=\; \dim(P_{\leq m}) - \dim(P_{\leq m} \cap \ker \chi) \, .$$

But proposition 7.1 states that $\ker \chi$ is the ideal in $\mathcal{O}(\mathbb{C}^4)$ generated by $z_1 z_4 - z_2 z_3 - 1$, and so it is clear that the linear map

$$P_{\leq m-2} \;\to\; P_{\leq m} \cap \ker \chi \qquad , \qquad Q(z) \mapsto (z_1 z_4 - z_2 z_3 - 1) \cdot Q(z)$$

is an isomorphism. Hence $\dim(P_{\leq m} \cap \ker \chi) = \dim(P_{\leq m-2})$, and $\dim \mathcal{H}_{poly} = \dim(P_m \oplus P_{m-1})$. But it's a well known combinatorial fact that $\dim P_m$ – the number of ways of choosing 4 non-negative integers whose sum is m – is $\binom{3+m}{3}$, and the result follows directly. $\qquad \square$

In practice, by looking at the asymptotics of the potential f, it is usually not difficult to compute the integer m.

Example 8.3. The lump metric on M studied in section 5 has a Kähler form $\pi^* \omega = \frac{i}{2} \partial \bar{\partial}(f \circ y)$, with $f(y) = \pi y \coth y$, and finite volume $\Omega = \pi^6/3$. From the asymptotics

$$f(y) = \pi y \left[1 + 2e^{-2y} + 2e^{-4y} + O(e^{-6y})\right] \qquad \text{as} \quad y \to +\infty$$

one gets that

$$(\cosh y)^l \, e^{-f(y)/2\hbar} \, \frac{\mathrm{d}}{\mathrm{d}y} [f'(y)]^3 \;=\; O(y \, e^{(-\pi/2\hbar + l - 2)y}) \qquad \text{as} \quad y \to +\infty \, ,$$

and so $m = \max\{l \in \mathbb{N} : l < 2 + \pi/2\hbar\}$.

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An interesting feature of this example is that, if we let $\hbar \to 0^+$, then $m \sim \pi/2\hbar$, and by corollary 8.1 we obtain

$$\dim_{\mathbb{C}} \mathcal{H}_{poly} \;\sim\; \frac{m^3}{3} \;\sim\; \frac{\pi^3}{24\hbar^3} = \frac{\Omega}{(2\pi\hbar)^3} \;.$$

This is exactly the answer expected in semi-classical quantum mechanics for the quantization of a phase space of volume Ω and real dimension 6 [8]. Before discussing the significance of this coincidence, we will first show that this property is more general, and is in fact valid for all the G-invariant Kähler metrics on M of finite volume.

Proposition 8.4. *Suppose $\omega = \frac{i}{2}\partial\bar{\partial}(f \circ y)$ is the Kähler form of a metric on M of finite volume Ω. Then the constant $m = m(f, \hbar)$ satisfies*

$$\frac{(3\Omega)^{1/3}}{2\pi\hbar} + k - 1 \;\leq\; m \;\leq\; \frac{(3\Omega)^{1/3}}{2\pi\hbar} + k \;,$$

where

$$k \;=\; k(f) \;:=\; \sup\left\{\lambda \in \mathbb{R}_0^+ : \int_0^{+\infty} e^{\lambda y}\, \frac{\mathrm{d}}{\mathrm{d}y}[f'(y)]^3\, \mathrm{d}y \ \text{is finite}\right\} \;\in\; [0, +\infty] \;.$$

Proof. By corollary 4.2 we have that

$$\Omega \;=\; \frac{\pi^3}{3} \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}y}[f'(y)]^3\, \mathrm{d}y \;=\; \frac{\pi^3}{3} \lim_{y\to+\infty} [f'(y)]^3 \;, \tag{25}$$

thus the finite volume condition implies that $k \geq 0$ and that $D := \lim_{y\to+\infty} f'(y)$ is a positive finite number. By L'Hôpital's rule we also get that $\lim_{y\to+\infty} y^{-1}f(y) = D$. It then follows that, for β real,

$$\lim_{y\to+\infty}\, (e^{-y} \cosh y)^l\, \exp\left\{\left[\beta - \frac{1}{2\hbar}\left(\frac{f(y)}{y} - D\right)\right] y\right\} \;=\; \begin{cases} +\infty \ \text{if } \beta > 0 \\ 0 \ \ \text{if } \beta < 0 \end{cases} \;. \tag{26}$$

Now, by definition, m is the biggest of the integers $l \in \mathbb{N}$ such that

$$\int_0^{+\infty} (\cosh y)^l\, e^{-f(y)/2\hbar}\, \frac{\mathrm{d}}{\mathrm{d}y}[f'(y)]^3\, \mathrm{d}y \tag{27}$$

converges. This integral, however, is the same as

$$\int_0^{+\infty} \left\{(e^{-y}\cosh y)^l \exp\left[\beta - \frac{1}{2\hbar}\left(\frac{f(y)}{y} - D\right)y\right]\right\} e^{(-\beta+l-D/2\hbar)\,y}\, \frac{\mathrm{d}}{\mathrm{d}y}[f'(y)]^3\, \mathrm{d}y \;.$$

If $l - D/2\hbar > k$, then choosing β in the interval $]0\,,\, l - D/2\hbar - k[$ and using (26) and the definition of k, it is clear that (27) diverges; thus $m \leq k + D/2\hbar$. If $l - D/2\hbar < k$, then choosing β in $]l - D/2\hbar - k\,,\, 0[$, the same arguments show that (27) converges; thus $m \geq k + D/2\hbar - 1$. Since from (25) we have $D = (3\Omega)^{1/3}/\pi$, the proposition is proved. $\qquad\square$

Corollary 8.5. *Given any G-invariant Kähler metric on M of finite volume Ω, let $\omega = \frac{i}{2} \partial \bar{\partial}(f \circ y)$ be its Kähler form. Then the associated quantum space $\mathcal{H}_{poly}$ is finite-dimensional iff $k(f)$ is finite. In this case $\dim_{\mathbb{C}} \mathcal{H}_{poly} \sim \Omega/(2\pi\hbar)^3$ as $\hbar \to 0^+$.*

Proof. It follows directly from corollary 8.2 and proposition 8.4. $\qquad\square$

Remark. The lump metric is an example with finite volume and finite-dimensional $\mathcal{H}_{poly}$. There are however metrics of finite volume with infinite dimensional $\mathcal{H}_{poly}$. For example, define $f'(t)$ as any extension of $t \mapsto (1 - e^{-t^2})$, $t \in [1, +\infty[$, to a smooth odd function on $\mathbb{R}$ with everywhere positive first derivative. Then calling f any primitive of f', the metric on M with Kähler potential $f \circ y$ has the desired property.

8.1 Discussion

The asymptotic value $\Omega/(2\pi\hbar)^3$ for the complex dimension of the quantum Hilbert space is both physically and geometrically quite significant. Physically because semi-classical statistical mechanics predicts that if you quantize a classical system with n degrees of freedom, thus with $2n$-dimensional phase space, you should get one independent quantum state of the system for each cell of volume $(2\pi\hbar)^n$ on the phase space [8]. Hence if the phase space has finite volume Ω, one gets $\Omega/(2\pi\hbar)^n$ independent quantum states.

The geometrical significance, on the other hand, arises from the fact that this asymptotic value is expected when the base manifold is compact, but not "a priori" for the non-compact $M = SL(2, \mathbb{C})$. The compact result is a direct consequence of the Hirzebruch-Riemann-Roch formula and the Kodaira vanishing theorem [5], and can be stated as follows.

On a compact Kähler $2n$-manifold of volume Ω, the Hilbert space of holomorphic quantization has finite complex dimension, and this grows asymptotically as $\Omega/(2\pi\hbar)^n$ when $\hbar \to 0^+$.

We are thus led to the following questions.

Question 1. Is there a version of the above result for the non-compact case ?

Our results suggest that there is, since $SL(2, \mathbb{C})$ is not compact and some of the G-invariant metrics for which corollary 8.5 holds – for example the lump metric – cannot be compactified.

Another example is the manifold $\mathbb{C}$ with any $U(1)$-invariant Kähler metric g. In this case the Kähler form also has a $U(1)$-invariant global potential – which we call ρ – and $\mathcal{H}_{poly}$ is the space of square-integrable complex polynomials on $\mathbb{C}$ with respect to the volume form $\exp(-\rho/2\hbar)\frac{i}{2}\partial\bar{\partial}\rho$. A simplified version of the method used in this paper then shows that, if the volume Ω of $(\mathbb{C}, g)$ and the dimension of $\mathcal{H}_{poly}$ are both finite, we also have

$$\dim_{\mathbb{C}} \mathcal{H}_{poly} \sim \Omega/(2\pi\hbar) \qquad \text{as} \;\; \hbar \to 0^+ \;.$$

In an optimistic spirit, we are thus led to formulate the following question.

Question 2. Let S be a closed complex submanifold of $\mathbb{C}^N$ (i.e. a Stein manifold) of complex dimension $n \leq N$, and let ω be any Kähler form on S. Since S is Stein, ω has a global potential $\rho \in C^\infty(S; \mathbb{R})$, and we can define

$$\mathcal{H}_{poly} := \left\{ \phi \in \mathbb{C}[z_1, \ldots, z_N] : \int_S |\phi|^2 e^{-\rho/2\hbar} \omega^n \; < \; +\infty \right\} .$$

Then, if the volume Ω of (M, ω) and the dimension of $\mathcal{H}_{poly}$ are both finite, is it always true that $\dim_{\mathbb{C}} \mathcal{H}_{poly} \sim \Omega/(2\pi\hbar)^n$ as $\hbar \to 0^+$?

Having in mind our examples, it is also possible that the result only holds for algebraic submanifolds of $\mathbb{C}^N$. Another point which would be worth to clarify is the relation between the spaces $\mathcal{H}_{poly}$ and $\mathcal{H}_{HQ}$.

Question 3. In our example of $SL(2, \mathbb{C})$ with a G-invariant Kähler metric of finite volume, is it true that whenever $\mathcal{H}_{poly}$ is finite-dimensional we have $\mathcal{H}_{poly} = \mathcal{H}_{HQ}$? And for more general Stein manifolds of finite volume ?

Recall that the finite-dimensionality of $\mathcal{H}_{poly}$ comes from the fact that the only polynomials on $\mathbb{C}^4$ which are integrable on $SL(2, \mathbb{C})$, are the ones of degree smaller than a certain constant. The above question asks if, in this case, the entire non-polynomial functions on $\mathbb{C}^4$ are automatically non-integrable on $SL(2, \mathbb{C})$. It is plausible that the answer is yes, since entire non-polynomial functions have very high growth rates in certain directions. However, if the answer is no, then perhaps in this case it is wiser to take $\mathcal{H}_{poly}$ as the quantum Hilbert space, instead of the traditional $\mathcal{H}_{HQ}$. The finite-dimensionality of $\mathcal{H}_{poly}$ ensures completeness and corollary 8.5 supports this choice.

8.2 Proof of proposition 8.1

The "only if" statement. We have to show that

$$\int_M |\phi|^2 e^{-(f \circ y)/2\hbar} \, \epsilon \;\; < \;\; +\infty \tag{28}$$

implies condition (24). Notice first that if (28) is satisfied, the "change of variables" theorem guarantees that for any of the G-action biholomorphisms $\psi_g : M \to M$,

$$\int_M |\phi|^2 e^{-(f \circ y)/2\hbar} \, \epsilon \;\; = \;\; \int_M |\phi \circ \psi_g|^2 e^{-(f \circ y)/2\hbar} \, \epsilon \, ,$$

where we have used the G-invariance of the volume form ϵ. So using the invariant (Haar) integral on G to average over the group, we get that

$$\int_M |\phi|^2 e^{-(f \circ y)/2\hbar} \, \epsilon \;\; = \;\; \int_M \left(\int_{g \in G} |\phi \circ \psi_g|^2 \right) e^{-(f \circ y)/2\hbar} \, \epsilon \, . \tag{29}$$

Now regard $\phi(z_1, \ldots, z_4)$ as a polynomial on $\mathbb{C}^4$, and write $\phi = \phi_0 + \cdots + \phi_l$, where ϕ_k is homogeneous of degree k. As in Appendix A, consider also the natural extension of the G-action ψ to the manifold $\mathbb{C}^4 \subset M$. We then have

$$\int_{g \in G} |\phi \circ \psi_g(z)|^2 \;=\; \sum_{k,j=0}^{l} \int_{g \in G} (\bar{\phi}_k \phi_j) \circ \psi_g(z) \;,$$

where each term of the sum is a smooth G-invariant function on $\mathbb{C}^4$. In particular, using the notation and proposition A.4 of Appendix A, each of these terms may be written as $F_{kj}(x(z), w(z))$, where $F_{kj} : \mathcal{B} \to \mathbb{C}$ is continuous, $x(z) = (|z_1|^2 + \cdots + |z_4|^2)/2$ and $w(z) = z_1 z_4 - z_2 z_3$. On the other hand, going back to the definition of ψ, we see that each component of $\psi_g(z)$ is just a linear combination of the components of z, and so in fact we must have

$$F_{kj}(x(z), w(z)) \;=\; \int_{g \in G} (\bar{\phi}_k \phi_j) \circ \psi_g(z) \;=\; \sum c_{i_1 \cdots i_k n_1 \cdots n_j} \, \bar{z}_{i_1} \cdots \bar{z}_{i_k} z_{n_1} \cdots z_{n_j} \;.$$

From this formula it is clear that, for any $\lambda \in \mathbb{R}_0^+$,

$$F_{kj}(\lambda^2 x(z), \lambda^2 w(z)) \;=\; F_{kj}(x(\lambda z), w(\lambda z)) \;=\; \lambda^{k+j} F_{kj}(x(z), w(z)) \;,$$

and in particular

$$x^{-l} \, F_{kj}(x, 1) \;=\; x^{-l+(k+j)/2} \, F_{kj}(1, x^{-1}) \;.$$

Since $k, j \leq l$, using the continuity of F_{kj} we then obtain that

$$\lim_{x \to +\infty} x^{-l} \, F_{kj}(x, 1) \;=\; \lim_{x \to +\infty} x^{-l+(k+j)/2} \, F_{kj}(1, x^{-1}) \;=\; \delta_{lk} \, \delta_{lj} \, F_{ll}(1, 0) \;.$$

Defining

$$h(x(z), w(z)) \;:=\; \int_{g \in G} |\phi \circ \psi_g(z)|^2 \;=\; \sum_{k,j=0}^{l} F_{kj}(x(z), w(z)) \;,$$

we therefore have that

$$\lim_{x \to +\infty} x^{-l} \, h(x, 1) \;=\; \sum_{k,j=0}^{l} \lim_{x \to +\infty} x^{-l} \, F_{kj}(x, 1) \;=\; F_{ll}(1, 0) \;. \tag{30}$$

Now, it will be shown in lemma 8.6 that $0 < F_{ll}(1, 0) < +\infty$, and so (30) implies that there is a constant $c > 0$ such that $h(x, 1) > c \, x^l$ for x big enough. On the other hand, using (29), proposition 4.1 of section 4, and that $w(z) = 1$ and $x(z) = \cosh(y(z))$ for $z \in M$, we have

$$\int_M |\phi|^2 \, e^{-(f \circ y)/2\hbar} \, \epsilon \;=\; \int_{z \in M} h(x(z), w(z)) \, e^{-(f \circ y)/2\hbar} \, \epsilon \;=$$

$$=\; \frac{\pi^3}{3(2\pi\hbar)^3} \int_0^{+\infty} h(\cosh y, 1) \, e^{-f(y)/2\hbar} \frac{\mathrm{d}}{\mathrm{d}y} [f'(y)]^3 \; \mathrm{d}y \;\geq$$

$$\geq\; \text{const.} + \frac{c}{24\hbar^3} \int_0^{+\infty} (\cosh y)^l \, e^{-f(y)/2\hbar} \frac{\mathrm{d}}{\mathrm{d}y} [f'(y)]^3 \; \mathrm{d}y \;,$$

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where const. is some finite real number. From this inequality it is clear that (28) implies condition (24) of proposition 8.1.

Lemma 8.6. *The constant $F_{ll}(1,0)$ is in $]0, +\infty[$.*

Proof. As we have seen above $F_{ll} : \mathcal{B} \to \mathbb{C}$ is continuous and, by definition,

$$F_{ll}(x(z), w(z)) = \int_{g \in G} |\phi|^2 \circ \psi_g(z) . \tag{31}$$

Since $(1,0) \in \mathcal{B}$ (see Appendix A), $F_{ll}(1,0)$ is a well-defined finite number, and from (31) it is clearly non-negative. Now call $\mathcal{V} := \{z \in \mathbb{C}^4 : z_1 z_4 - z_2 z_3 = 0\}$, and let q be any point in $\mathcal{V} \setminus \{0\}$. Since $q' := x(q)^{-1/2} q \in \mathcal{V}$ and $x(q') = 1$, we have

$$\int_{g \in G} |\phi_l \circ \psi_g(q')|^2 = F_{ll}(x(q'), w(q')) = F_{ll}(1,0) .$$

Hence, if $F_{ll}(1,0) = 0$, we get that $\phi_l \circ \psi_g(q') = 0$ for any $g \in G$, and in particular $\phi_l(q') = 0$. But ϕ_l is homogeneous, and so also $\phi_l(q) = 0$. From the arbitrariness of q it follows that ϕ_l vanishes on $\mathcal{V}$ – the zero set of the irreducible polynomial $z_1 z_4 - z_2 z_3$. Finally from Hilbert's Nullstellensatz we conclude that ϕ_l is divisible by $z_1 z_4 - z_2 z_3$. This contradicts the hypothesis of proposition 8.1, and therefore $F_{ll}(1,0) > 0$. $\qquad\square$

The "if" statement. As before, write $\phi = \phi_0 + \cdots + \phi_l$. Then $|\phi|^2 \leq |\phi_0|^2 + \cdots + |\phi_l|^2$. Since $x(z) = (|z_1|^2 + \cdots + |z_4|^2)/2$, we have that $|z_k| \leq \sqrt{2x(z)}$ for all z. But ϕ_j is homogeneous of degree $j \leq l$, thus

$$|\phi_j(z)|^2 \leq c_j (2x(z))^j \leq c_j (2x(z))^l , \quad z \in \mathbb{C}^4 ,$$

for some positive constants c_j. Calling $c = \sum_{j=0}^l c_j$, using proposition 4.1 of section 4 and that $x(z) = \cosh y(z)$ for $z \in M$, we finally get

$$\int_M |\phi|^2 e^{-(f \circ y)/2\hbar} \epsilon \leq \sum_{j=0}^l \int_M |\phi_j|^2 e^{-(f \circ y)/2\hbar} \epsilon \leq \int_{z \in M} c (2x(z))^l e^{-(f \circ y)/2\hbar} \epsilon =$$

$$= \frac{c \, 2^l \, \pi^3}{3(2\pi\hbar)^3} \int_0^{+\infty} (\cosh y)^l \, e^{-f(y)/2\hbar} \frac{\mathrm{d}}{\mathrm{d}y} [f'(y)]^3 \, \mathrm{d}y .$$

From this inequality it is clear that condition (24) of proposition 8.1 implies (28), and so $\phi \in \mathcal{H}_{poly}$. $\qquad\square$

Appendix A

In this appendix we study the action ψ of the group $G := SU(n) \times SU(n)$ on the manifold $M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$ of complex $n \times n$ matrices defined by

$$\psi : G \times M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C}) \quad , \quad (U_1, U_2, A) \mapsto U_1 A U_2^{-1} . \qquad (32)$$

The results obtained are used in sections 2 and 8.

According to [7, p. 396] every matrix $M \in GL(n, \mathbb{C})$ can be decomposed in the form $M = KAK'$, where $K, K' \in U(n)$ and A is real diagonal with positive entries in the diagonal. Notice that multiplying K and K' by permutation matrices, if necessary, we may assume that the diagonal entries of A do not decrease with the row index.

Lemma A.1. *Every matrix $M \in M(n, \mathbb{C})$ may be decomposed in the form $M = U_1 A U_2 e^{i\theta}$, where $U_1, U_2 \in SU(n)$, $\theta \in \mathbb{R}$, and A is a real diagonal matrix with non-negative diagonal entries which do not decrease with the row index.*

Proof. Given $M \in M(n, \mathbb{C})$ there is a sequence $\{M_j\}$ in $GL(n, \mathbb{C})$ with $M_j \rightarrow M$. Using the decomposition described above, for each M_j we have

$$M_j = K_j A_j K'_j .$$

Since the sequences $\{K_j\}$ and $\{K'_j\}$ are in the compact group $U(n)$, there are convergent subsequences $K_{j_l} \rightarrow K$ and $K'_{j_l} \rightarrow K'$ when $l \rightarrow +\infty$, where $K, K' \in U(n)$. Defining

$$A := K^\dagger M (K')^\dagger = \lim_{l \rightarrow +\infty} (K_{j_l})^\dagger M_{j_l} (K'_{j_l})^\dagger = \lim_{l \rightarrow +\infty} A_{j_l} ,$$

the fact that A_{j_l} is diagonal with positive ordered diagonal entries, implies that A is diagonal with non-negative ordered diagonal entries; furthermore $KAK' = M$. Since $K, K' \in U(n)$, they can always be written as matrices in $SU(n)$ times a phase, and this ends the proof. $\square$

We will now find functions on $M(n, \mathbb{C})$ which separate the orbits of ψ, and hence can be used as coordinates in the space of orbits. For this define the polynomials P_j on $M(n, \mathbb{C})$ by

$$\det(B + \lambda I) = \sum_{j=0}^{n} \lambda^j P_j(B) .$$

We then have:

Proposition A.2. *Two matrices $M, N \in M(n, \mathbb{C})$ lie in the same orbit of ψ if and only if $P_j(M^\dagger M) = P_j(N^\dagger N)$ for $1 \leq j \leq n$ and $\det N = \det M$.*

Proof. If N and M are in the same orbit, i.e. $N = U_1 M U_2$ for some $U_1, U_2 \in SU(n)$, then $N^\dagger N = U_2^\dagger M^\dagger M U_2$, and the stated conditions are clearly satisfied.

Conversely, suppose that $P_j(N^\dagger N) = P_j(M^\dagger M)$ for $1 \le j \le n$ and $\det M = \det N$. Then $N^\dagger N$ and $M^\dagger M$ have the same characteristic polynomial, and hence the same eigenvalues. On the other hand, from lemma A.1 we have the decompositions

$$M = U' \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \, U e^{i\theta} \quad \text{and} \quad N = \tilde{U}' \operatorname{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \, \tilde{U} \, e^{i\beta} \,, \qquad (33)$$

so

$$M^\dagger M = U^\dagger \operatorname{diag}(\lambda_1^2, \ldots, \lambda_n^2) \, U \quad \text{and} \quad N^\dagger N = \tilde{U}^\dagger \operatorname{diag}(\tilde{\lambda}_1^2, \ldots, \tilde{\lambda}_n^2) \, \tilde{U}$$

have eigenvalues $\{\lambda_1^2, \ldots, \lambda_n^2\}$ and $\{\tilde{\lambda}_1^2, \ldots, \tilde{\lambda}_n^2\}$, respectively. Since lemma A.1 also guarantees that the $\lambda_j, \tilde{\lambda}_j$ are non-negative and ordered, we conclude that $\lambda_j = \tilde{\lambda}_j$ for $1 \le j \le n$. Hence

$$N = \tilde{U}'(U')^{-1} M U^{-1} \tilde{U} \, e^{i(\beta - \theta)} \,.$$

If $\det M = \det N \ne 0$, then taking the determinant of the above equation we get that $\det(e^{i(\beta - \theta)} I) = 1$, and so $e^{i(\beta - \theta)} I$ is in $SU(n)$. This shows that $N = U_1 M U_2$ for some $U_1, U_2 \in SU(n)$.

If $\det M = \det N = 0$, then (33) implies that the product of the λ_j is zero, therefore $\lambda_1 = \tilde{\lambda}_1 = 0$, because the λ_j are non-negative and ordered. Defining

$$\Lambda := \operatorname{diag}(e^{i(\theta - \beta)(n-1)}, e^{i(\beta - \theta)}, \ldots, e^{i(\beta - \theta)}) \quad \in \; SU(n)$$

we then get

$$N = \tilde{U}' \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \, \tilde{U} \, e^{i\beta} = \tilde{U}' \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \, \Lambda \tilde{U} \, e^{i\theta} = \tilde{U}'(U')^{-1} M U^{-1} \Lambda \tilde{U} \,,$$

which shows that, also in this case, $N = U_1 M U_2$ for some $U_1, U_2 \in SU(n)$. $\qquad\square$

These results are now going to be used in the study of G-invariant functions on $M(2, \mathbb{C})$ and $SL(2, \mathbb{C})$. Define the smooth map

$$\beta : M(2, \mathbb{C}) \simeq \mathbb{C}^4 \to \mathbb{R} \times \mathbb{C} \quad, \quad \beta(z) = (x(z), w(z)) \,,$$

where

$$x(z) = \frac{1}{2}(|z_1|^2 + \cdots + |z_4|^2) \quad \text{and} \quad w(z) = z_1 z_4 - z_2 z_3 \,.$$

It follows from the proposition above that two points in $M(2, \mathbb{C})$ lie in the same ψ-orbit iff they have the same image by β. In particular any G-invariant function $\tilde{h}$ on $M(2, \mathbb{C})$ may be written $\tilde{h} = h \circ \beta$, where h is some function defined on the image of β. We will now show that the continuity of $\tilde{h}$ implies the continuity of h – a result used in section 8.

Lemma A.3. *The image of β is $\mathcal{B} := \{(a, u) \in \mathbb{R} \times \mathbb{C} : a \ge |u|\}$.*

Proof. From the identity

$$x(z)^2 - |w(z)|^2 = \frac{1}{4}(|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)^2 + |z_1\bar{z}_3 + z_2\bar{z}_4|^2 \geq 0$$

it follows that $x(z) \geq |w(z)|$, thus the image of β is contained in $\mathcal{B}$.

Conversely, defining $g : \mathcal{B} \to \mathbb{C}^4$ by

$$g(a,u) = \left(u(a + \sqrt{a^2 - |u|^2})^{-1/2}, 0, 0, (a + \sqrt{a^2 - |u|^2})^{1/2} \right) , \tag{34}$$

one can easily check that $\beta \circ g(a,u) = (a,u)$, and so $\mathcal{B}$ contains the image of β. $\square$

Proposition A.4. *Let X be a topological space, $\mathcal{V}$ a subset of the image of β, and $h : \mathcal{V} \to X$ a map such that $h \circ \beta$ is continuous on $\beta^{-1}(\mathcal{V})$. Then h is continuous.*

Proof. Consider the map $g : \mathcal{B} \to \mathbb{C}^4$ defined in (34). This map is clearly continuous on $\mathcal{B} \setminus \{(0,0)\}$ and, for (a,u) approaching the origin from this set,

$$\lim_{(a,u)\to(0,0)} \left| \frac{u}{\sqrt{a + \sqrt{a^2 - |u|^2}}} \right| \leq \lim_{(a,u)\to(0,0)} \frac{a}{\sqrt{a}} = 0 .$$

Thus g is also continuous at $(0,0)$ and vanishes at this point. Finally, since

$$h(a,u) = (h \circ \beta) \circ g(a,u) \quad \text{for all} \ (a,u) \in \mathcal{V} ,$$

we conclude that the continuity of $h \circ \beta$ implies the continuity of h. $\square$

Now suppose we restrict the action ψ of G to the submanifold $SL(2,\mathbb{C}) \subset M(2,\mathbb{C})$. Since the function $w(z)$ is identically 1 on $SL(2,\mathbb{C})$, we have that two points in this submanifold lie in the same ψ-orbit iff they have the same image by $x(z)$. From lemma A.3 it follows that $x(SL(2,\mathbb{C})) = [1,+\infty[$, and since $\cosh^{-1}$ is injective on this interval, we have that $y := \cosh^{-1} \circ x$ also separates orbits in $SL(2,\mathbb{C})$. We can now prove proposition 2.1 of section 2.

Proof of proposition 2.1. From the paragraph above, it is clear that any smooth G-invariant function $\tilde{f}$ on $SL(2,\mathbb{C})$ may be written as $\tilde{f} = f \circ y$, for some unique $f : [0,+\infty[\to \mathbb{R}$. Now consider the smooth map $h : \mathbb{R} \to SL(2,\mathbb{C})$ defined by

$$h(t) = \begin{bmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{bmatrix} .$$

One can easily check that $y \circ h(t) = t$ for $t \geq 0$, hence

$$f(t) = (f \circ y) \circ h(t) = \tilde{f} \circ h(t) , \quad t \geq 0 ,$$

which implies that f is smooth. Note also that h is defined on $\mathbb{R}$, and from the G-invariance of $\tilde{f}$ we get $\tilde{f} \circ h(-t) = \tilde{f} \circ h(t)$; thus f can be extended to an even function on $\mathbb{R}$. $\square$

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